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Existence of Global Solutions H^s with Small Initial Conditions

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Abstract: The Schrodinger equation is one of the most fundamental equations in quantum mechanical theory. Since the appearance of this equation, there has been a large number of works studying its properties. Previously, most of the research focused on the linear Schrodinger equation. Recently, a number of experts such as T. Kato, T. Tao, C. Kening have focused on studying nonlinear Schrodinger equations. The paper is to introduce the work of T. Kato, one of the important works in this research direction.

Keywords: Schodinger equation, linear Schodinger equation, nonlinear Schodinger equation (NLSE), global solution.

I. Introduction

We consider an initial value problem of nonlinear Schrodinger equation (NLSE)

$$\partial_t u = i(\Delta u - F(u)),\ t \geq 0,\ x \in \square^m,\ m \in \square .$$

with the following asumptions of F(u)

(1) $F \in C^1(\square, \square);\ F(0) = 0,$

(2) $DF(\xi) = O(|\xi|^{k-1})$ with $k \geq 1$, as $|\xi| \to \infty$.

Theorem A.in (2, we suppose $k < 1 + \dfrac{4}{m-2}$

(i) for any $\phi \in H^1 = H^1(\square^m)$, provided with $T > 0$ and an unique solution $u \in C([0, T); H^1)$ of (NLSE) with $u(0) = \phi$.

(ii) u has some addionalt properties

$$\partial u \in L^r([0, T); L^q) \text{ with } \frac{1}{q} + \frac{2}{mr} = 1,\ \frac{1}{2} - \frac{1}{m} < \frac{1}{q} < \frac{1}{2}.$$

Theorem B. Given $k < 1 + \dfrac{4}{m}$. For any $\phi \in L^2$ then $T > 0$ and an unique solution u of (NLSE) with the following properties

(i) $u \in C([0, T); L^2)$ with $u(0) = \phi$,

(ii) $u \in L^r((0, T); L^q)$ with $\dfrac{1}{q} + \dfrac{2}{mr} = 1,\ \dfrac{1}{2} - \dfrac{1}{m} < \dfrac{1}{q} < \dfrac{1}{2}$.

II. Premilinaries

In this section we will prove the existence of H^s – gobal solution under slighter conditions (see [3]). In addition, we need an unique condition as follow

(F3) $F(\xi) = 0^{\{s\}}(|\xi|^k)$ khi $|\xi| \to 0$, where $h = 1 + \frac{4}{m} = \chi(0)$.

Theorem 2.1. Assume (F1), (F2), (F3)

(i) If $\phi \in H^s$ with $\|\phi\|_{H^\sigma}$ is small enough, then (NLSE) has a unique global solution u in $BC([0, \infty); H^s) \cap \Psi_s$ with $u(0) = \phi$, where Ψ_s is in $[0; \infty)$, where σ is an arbitrary number calculated with $s \leq \dfrac{m}{2}$ or $s > \dfrac{m}{2}$.

ii) If $s \leq \frac{m}{2}$ and F are homogeneous of order k, then the result is correct for $\phi \in H^s$ with $\| \Lambda^{\bar{\sigma}} \phi \|_2$ small enough, where $\bar{\sigma} = 0 \vee \chi^{-1}(k)$.

Note

(a) Recall that we assume{s} $\leq$ k unless F is a polynomial

(b) If s $\leq \frac{m}{2}$, then (i) requires that $\|\phi\|_{H^s}$ is small; therefore the result is optimal at least when $\sigma = \bar{\sigma} = 0 \vee \chi^{-1}(k)$, the minimum positive value of σ.

If s $> \frac{m}{2}$, it seems to have no optimal values. Otherwise, (ii) contains $\|\Lambda^{\sigma}\phi\|_2$ and $\sigma = \bar{\sigma}$. Note that if k = χ(s) (cristical case) having only one value of σ is s. Of course, all of small conditions are valid if $\|\phi\|_{H^s}$ is enough small.

(c) The scattering theory can be developed to the small solutions u. For instance, let $\phi_{\pm} \in H^s$ be satisfied by u(t) – U(t)$\phi_{\pm} \rightarrow 0$ in H^s when t $\rightarrow \pm\infty$.

Lemma 2.1. If $\phi \in H^s$ along with $\|\phi\|_2 \vee \|\Lambda^{\sigma}\phi\|_2$ is enough small, then parameters L, K, N can be chosen such that mapping Φ from Ξ to Ξ is to be contraction mapping in L(P) – metric.

Proof:

Case (i") (s $\leq \frac{m}{2}$ và h > {s})

It is always available if s $\in$ (0, 1) rather than m = 1. So F = $F_{\{s\}} + F_k$. If k > {s} then F_k is 0 simutaneously nearby $\xi = 0$ such that $F_{\{s\}} = O(|\xi|^h)$ with ξ small. Since {s} < h, it is still true for large ξ. It is convenient to rewrite $F_{\{s\}}$ as F_h such that F_h and F_k are homogeneous degree of h and k, respectively.

Let R $\in l'$, Q_h = P $\in l$ and $Q_k \in$ (OP] on the main diagonal with

$$R = hP = P + (k-1)Q_k, \quad Q_k \in [PQ], \quad Q \in l_{\sigma} \cap (OP],$$

where Q_k is not necessary if k $\leq$ h.

Case (i"') (s $> \frac{m}{2}$)

For any fix point $\sigma \in \left(\frac{m}{2}, s\right)$, we have the set Ξ so that u $\in \Xi$ is bounded by cL $\vee$ N. Therefore, we can change the values of F(ξ) with $|\xi|$ > cL $\vee$ N and F is finite increasing k > {s}. The Sobolev's theorem stated that $\|u: Q\| \leq L \vee N$ with Q $\in$ (OP], it is valid for all Q_j.

Case (ii)

If F is homogeneous, then let F_k = F, where h $\leq$ k $\leq \chi$(s) is just a unique point Q_j as Q_k. If $\sigma = \bar{\sigma} = \chi^{-1}(k)$, we get $\|u: Q_k\| \leq cN$, since then ψ_2(L, N) depends really on N.

III. Main Results

Theorem 3.1. Assuming that(F_1), (F_2), for any $\phi \in H^s$, include T > 0 and a unique solution u $\in C([0, T]; H^s) \cap \Psi_s$ of (NLS) subject to u(0) = ϕ. The complement space Ψ_s is negligible because (NLSE) is absolute adjustment in H^s if s $\geq \frac{m}{2}$, or if s $< \frac{m}{2}$ and

$$k < 1 + \frac{4 \wedge (2s+s)}{m-2s} \qquad \left(k \leq \frac{2}{1-2s} \text{ if } m = 1\right).$$

Theorem 3.2. With the asumptions of theorem 3.1, provided that v is a solution of (NLSE) such that

$$v \in C([0, T]; H^s) \cap L^r_{loc}((0,T); L^{q,s})$$

for a single pair of (q, r) $\neq$ (2, ∞) as in (A). Then, v is a uniquely canonical solution.

Lemma: With the asumptions of theorem 3.1, let u be H^s – a canonical solution of (NLSE) on a finite interval [0, T]. Let $\{\phi_n\}$ be a sequence satifying $\phi_n \to \phi = u(0)$ in H^s. Then, for T' > 0, u_n with $u_n(0) = \phi_n$ exists in [0, T'] with a large n , and $u_n \to u$ in $C([0, T']; H^{s'})$ when $n \to \infty$, for any s' < s. If s = 0 or 1, T' = T, s' = s can be chosen.

Proof:

v is a canonical soltion on interval [τ, T – τ) provided with τ> 0 and small, since $v \in L^r_{loc}((0,T); L^{q,s})$ along with $v \in L^r((\tau, T-\tau); L^{q,s})$. If $v_\tau(t) = v(t + \tau)$ and $0 < \tau < \frac{T}{4}$ then $v(\tau)$ is a canonical solution in $[0, \frac{T}{2}]$ subject to $v_\tau(0) = v(\tau)$.

Let u be H^s – a canonical solution subject to u(0)= v(0); we will prove that v = u. Since $v_\tau(0) = v(\tau) \to v(0) = u(0)$ in H^s when $\tau \to 0$, it was leaded by the lemma that $v_\tau \to u$ in $C([0, T']; L^2)$ and T' > 0. Since $v_\tau \to v$ in $C([0, \frac{T}{2}]; H^s)$. In other words, we proved that v = u in $[0, T' \wedge \frac{T}{2}]$.

IV. Conclusion

In this paper, we presented the nonlinear Schodinger equation (NLSE) with the general potential function F(u) that there is not any conservative rule. The main asumption is an increasing speed of F(u) as well as $|u|^k$ when |u| is large, in addition, the smooth of F depends on a problem considered. The unique theorem was proved with the asumption of the smallest smooth of F and u. The theorem of existence of local solution in space H^s was proved by using a complement space that is Lebesgue space. We assumed that $k \le 1 + \frac{4}{m-2s}$ if $s < \frac{m}{2}$; $k < \infty$ if $s = \frac{m}{2}$ ($s > \frac{m}{2}$ is not necessary). Furthermore, The theorem of general existence was proved for H^s – a global soltion subject to initial condition that is small, and an additional condition is $F(u) = O(|u|^{1+4/m})$. It is available for all values $s \ge 0$ if $m \le 6$, but there are still some difficulties if $m \ge 7$ and if F(u) is not a polynomial of u and $\bar{u}$.

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