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Analytical solution of 2D SPL heat conduction model

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ABSTRACT : The heat transport at microscale is vital important in the field of micro-technology. In this paper heat transport in a two-dimensional thin plate based on single-phase-lagging (SPL) heat conduction model is investigated. The solution was obtained with the help of superposition techniques and solution structure theorem. The effect of internal heat source on temperature profile is studied by utilizing the solution structure theorem. The whole analysis is presented in dimensionless form. A numerical example of particular interest has been studied and discussed in details.
KEYWORDS – SPL heat conduction model, superposition technique, solution structure theorem, internal heat source

1. INTRODUCTION

Cattaneo [1] and Vernotte [2] removed the deficiency [3]-[6] occurs in the classical heat conduction equation based on Fourier's law and independently proposed a modified version of heat conduction equation by adding a relaxation term to represent the lagging behavior of energy transport within the solid, which takes the form

$$\tau_q \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -k\nabla T \tag{1}$$

where k is the thermal conductivity of medium and τ_q is a material property called the relaxation time. This model characterizes the combined diffusion and wave like behavior of heat conduction and predicts a finite speed

$$c = \left(\frac{k}{\rho c_b \tau_q} \right)^{\frac{1}{2}} \tag{2}$$

for heat propagation [7], where ρ is the density and c_b is the specific heat capacity. This model addresses short time scale effects over a spatial macroscale. Detailed reviews of thermal relaxation in wave theory of heat propagation were performed by Joseph and Preziosi [8], and Ozisik and Tzou [9]. The natural extension of CV model is

$$\mathbf{q}(\mathbf{r}, t+\tau_q) = -k\nabla T(\mathbf{r}, t) \tag{3}$$

which is called the single-phase-lagging (SPL) heat conduction model [10]-[14]. According to SPL heat conduction model, there is a finite built-up time τ_q for onset of heat flux at $\mathbf{r}$, after a temperature gradient is imposed there i.e. τ_q represents the time lag needed to establish the heat flux (the result) when a temperature gradient (the cause) is suddenly imposed.

Due to the complexity of the SPL model, the exact solution can be obtained only for specific initial and boundary conditions. The most popular solution methodology has resorted to either finite-difference or finite-element methods. Only a few simple cases can be solved analytically. In the literature most popular analytical solutions are the method of Laplace transformation [15], Fourier solution technique [16], Green's function solution [17], and the integral equation method by Wu [18] for the solution of the hyperbolic heat conduction equation.

Recently, Lam and Fong [19] and Lam [20] conducted studies by employing the superposition technique along with solution structure theorems for the analysis of the CV hyperbolic heat conduction equation and one dimensional generalized heat conduction model. The temperature profile inside a one-dimensional region was obtained in the form of a series solution. The method is relatively simple and requires only a basic background in applied mathematics. However, it was noted that solution structure theorems concentrated only on physical problems subjected to homogeneous boundary conditions. It was pointed out that there is a way to solve problems with non-homogeneous boundary conditions by performing appropriate functional transformations, namely by using auxiliary functions.

The purpose of this study is to apply solution structure theorems to study two dimensional SPL heat conduction in a finite plate subjected to homogeneous boundary conditions. The SPL heat conduction equation is solved using the superposition principle in conjunction with solution structure theorems. The outline of the paper is as follows. SPL heat conduction model is given in section 2. Section 3 deals solution of single-phase-lagging heat conduction model. Section 4 contains result and discussion. Conclusion is given in section 6.

2. 2D SPL HEAT CONDUCTION MODEL

The combination of Fourier's law of heat conduction

$$q = -k\frac{\partial T}{\partial y} \tag{4}$$

and law of conservation of energy [21]

$$\rho c_b \frac{\partial T}{\partial t} = -\frac{\partial q}{\partial y} + g^* \tag{5}$$

provides the law of heat conduction as follows

$$\rho c_b \frac{\partial T}{\partial t} = k\nabla^2 T + g^* \tag{6}$$

where g^* denotes the internal energy generation rate per unit volume inside the medium. In two dimension (6) can be written as

$$\rho c_b \frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^{*2}} + \frac{\partial^2 T}{\partial y^{*2}}\right) + g^* \tag{7}$$

The above (7) is the classical diffusion model which governs thermal energy transport in solids. By introducing dimensionless parameters $\theta = \frac{kcT}{\alpha f_r}, x = \frac{cx^*}{2\alpha}, y = \frac{cy^*}{2\alpha}, F_0 = \left(\frac{c^2}{2\alpha}\right)t$. Equation (7) can be expressed in dimensionless form as

$$2\frac{\partial \theta}{\partial F_0} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + g \tag{8}$$

where Fourier number F_0 represents dimensionless time. The CV constitutive relation (1) together with the energy conservation (5) gives the equation governing propagation of thermal energy

$$\frac{\partial T}{\partial t} + \tau_q \frac{\partial^2 T}{\partial t^2} = \alpha\left(\frac{\partial^2 T}{\partial x^{*2}} + \frac{\partial^2 T}{\partial y^{*2}}\right) + \frac{\alpha}{k}\left(g^* + \tau_q \frac{\partial g^*}{\partial t}\right) \tag{9}$$

where α is the thermal diffusivity of the material and the relaxation time $\tau_q = \alpha / c^2$. On the left hand side of above equation, the second order time derivative term indicates that heat propagates as a wave with a characteristic speed given by (2) and the first order time derivative corresponds to a diffusive process, which damps spatially the heat wave. One can see that if energy travels at an infinite propagation speed (i.e. $c \to \infty$), then (9) reduces to the two dimensional parabolic heat conduction equation (based on Fourier law). The (9) can be expressed in dimensionless form as

$$2\frac{\partial \theta}{\partial F_0} + \frac{\partial^2 \theta}{\partial F_0^{\,2}} = \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right) + \left(g + \frac{1}{2}\frac{\partial g}{\partial F_0}\right) \tag{10}$$

The above (10) can be written in simplified form as

$$2\frac{\partial\theta}{\partial F_0}+\frac{\partial^2\theta}{\partial F_0{}^2}=\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}\right)+G \tag{11}$$

In present study, an isotropic thin plate, $0\le x,y\le 1,$ with uniform thickness and constant thermo-physical properties, is assumed. Initially, the thin plate is at temperature $\theta(x,y,0)=\theta_2,$ which is a function of positions within the thin plate and rate of change in temperature is θ_3. For time $F_0>0,$ the following boundary conditions will be considered

$$\theta(x,y,0)=\theta_2,\frac{\partial\theta(x,y,0)}{\partial F_0}=\theta_3 \tag{12}$$

$$\theta(0,y,F_0)=0,\theta(1,y,F_0)=0 \tag{13}$$

$$\theta(x,0,F_0)=0,\theta(x,1,F_0)=0 \tag{14}$$

3. SOLUTION

The superposition technique can be applied to solve linear heat transfer problem with non-homogeneous term [7, 22, 23]. With the application of superposition principle, the original problem (11) can be divided into three sub-problems by setting initial conditions and $\left(G(x,y,F_0)\right)$ as (1) $G=\theta_2=0,$ (2) $G=\theta_3=0,$ and (3) $\theta_2=\theta_3=0$. Solution to these sub-problems is designated as S_1,S_2,S_3. Therefore, the general solution of the original hyperbolic SPL heat conduction model is $S=S_1+S_2+S_3$.

3.1. Solution Structure Theorem

With the help of solution structure theorem [7], once the solution of sub-problem (1) is known, solution of sub-problems (2) and (3) can be obtained as follows

$$S_2=\left(2+\frac{\partial}{\partial F_0}\right)\mathrm{F}\left(x,y,F_0,\theta_2\right)+B\mathrm{F}\left(x,y,F_0,\lambda_{m,n}\theta_2\right)$$

$$S_3=\int_0^{F_0}\mathrm{F}\left(x,y,F_0-\tau,G(x,y,\tau)\right)d\tau$$

where $\mathrm{F}\left(x,y,F_0,\theta_3\right)$ be the solution of sub-problem (1).

3.2. Solution of 2D-SPL Heat Conduction Model

This section only devoted to the solution of the sub-problem (1) of SPL heat conduction model. For the given initial and boundary conditions, one can write solution to the governing equation by using Fourier series as

$$\theta(x,y,F_0)=\sum_{m,n}^{\infty}\theta_{m,n}(F_0)Cos(\lambda_m x)Cos(\lambda_n y) \tag{15}$$

By substituting above (15) into (11) and after some manipulation we get following

$$\frac{\partial^2\theta_{m,n}}{\partial F_0^2}+2\frac{\partial\theta_{m,n}}{\partial F_0}+\lambda_{m,n}\theta_{m,n}=0 \tag{16}$$

The Solution of above takes the form

$$\theta_{m,n}(F_0)=e^{\alpha_{m,n}F_0}\left\{a_{m,n}Sin(\beta_{m,n}F_0)+b_{m,n}Cos(\beta_{m,n}F_0)\right\} \tag{17}$$

where $\alpha_{m,n}$ and $\beta_{m,n}$ are defined as follows

$$\alpha_{m,n}=-1,\beta_{m,n}=\sqrt{\lambda_{m,n}-1},\ \lambda_{m,n}=\lambda_m^2+\lambda_n^2;\lambda_m=m\pi,\lambda_n=n\pi\,.$$

By substituting above (17) into (15) solution of the sub-problem (1) can be expressed as follows

$$S_1 \equiv \theta(x,y,F_0) = \sum_{m,n}^{\infty} e^{\alpha_{m,n}F_0} \left\{ a_{m,n} Sin(\beta_{m,n}F_0) + b_{m,n} Cos(\beta_{m,n}F_0) \right\} \times Cos(\lambda_m x) Cos(\lambda_n y) \qquad (18)$$

Now to find the coefficients $a_{m,n}$ and $b_{m,n}$ we consider initial conditions $\theta_2 = 0$, then $b_{m,n} = 0$ and $a_{m,n}$ may be obtained as

$$a_{m,n} = \frac{2}{\beta_{m,n}} \int_0^1 \int_0^1 \theta_3 Cos(\lambda_m x) Cos(\lambda_n y) dxdy \, .$$

Hence the solution of the problem is complete for $m,n > 0$. Since the solution contains *Cosine* terms at the end of (18), therefore for $m,n = 0$ there is also a solution of the problem. For $m,n = 0$, (16) becomes

$$\frac{\partial^2 \theta_0}{\partial F_0^2} + 2\frac{\partial \theta_0}{\partial F_0} = 0$$

With the application of initial conditions, solution of above is

$$\theta_0(x,y,F_0) = \frac{1}{2}(1 - e^{-2F_0})\theta_3 \qquad (19)$$

Thus the final solution of the two dimensional SPL heat conduction model is

$$\theta(x,y,F_0) = \theta_{m,n}(x,y,F_0) + \theta_0(x,y,F_0) = \frac{1}{2}(1 - e^{-F_0})\theta_3 + 2 \times$$

$$\sum_{m,n=1}^{\infty} \frac{e^{\alpha_{m,n}F_0}}{\beta_{m,n}} \int_0^1 \int_0^1 \theta_3(\xi,\psi) Cos(\lambda_m \xi) Cos(\lambda_n \psi) d\xi d\psi Sin(\beta_{m,n}F_0) Cos(\lambda_m x) Cos(\lambda_n y) \qquad (20)$$

4. RESULTS AND DISCUSSION

This section presents complete solution of two dimensional SPL heat conduction model under different initial and boundary conditions. By utilizing the solution structure theorem, the effect of internal heat source on temperature profile has been studied and is given in case 2. The figures presented in this study, only the parameters whose values different from the reference value are indicated.

Case 1: $\theta_2 = 0, \theta_3 = Sin(xy), G = 0$.

In this case in the absence of internal heat source, effect of Fourier number has been observed. Figs. 1-2 present the spatial temperature profile for two Fourier number $F_0 = 0.5$ and 1.0 . The dimensionless temperature firstly increases with F_0 as Fourier number is a measure of rate of heat conduction with the heat storage in a given volume element. Larger the Fourier number, deeper is the penetration of heat into the body over a given period of time.

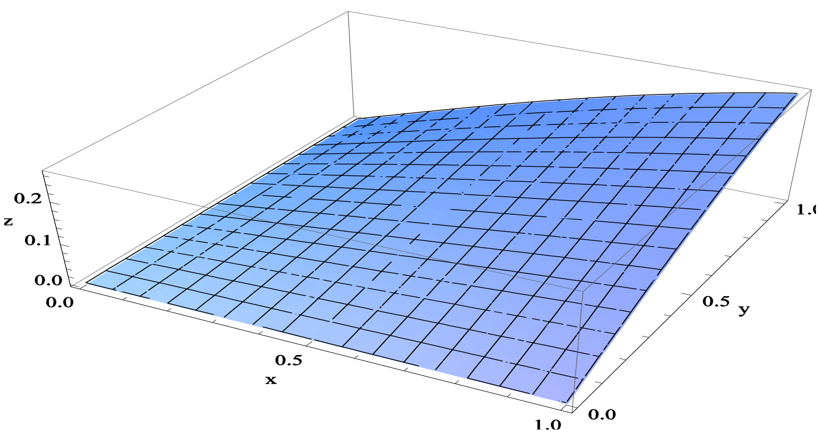

Fig. 1 Spatial temperature profile at $F_0 = 0.5$.

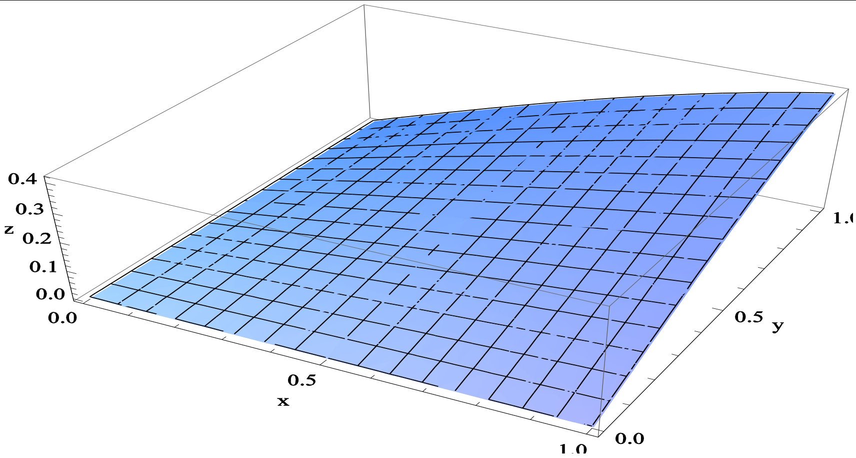

Fig. 2 Spatial temperature profile at $F_0 = 2.0$.

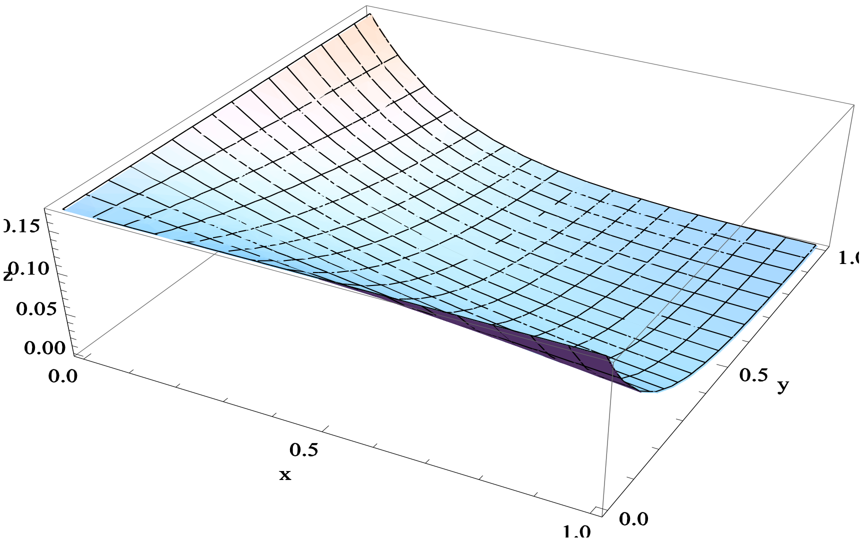

Fig. 3 Spatial temperature profile at $F_0 = 1.0, \delta = 0.5, \mu = 5$.

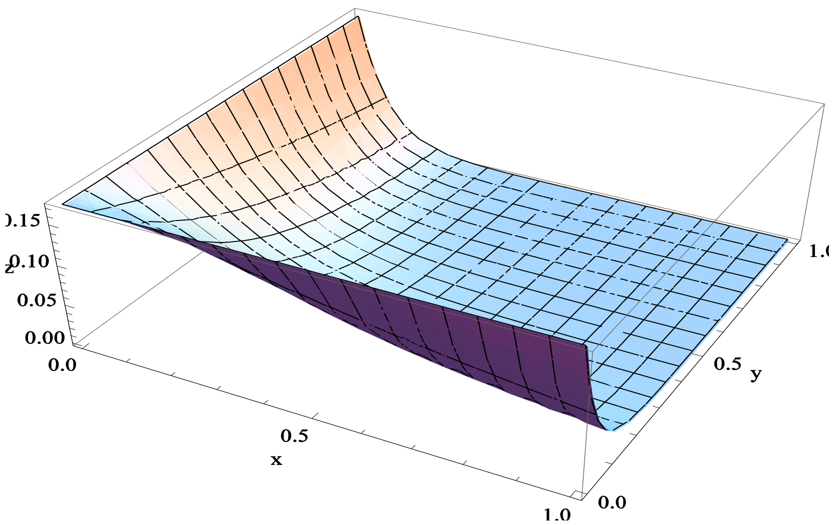

Fig. 4 Spatial temperature profile at $F_0 = 1.0, \delta = 0.5, \mu = 15$.

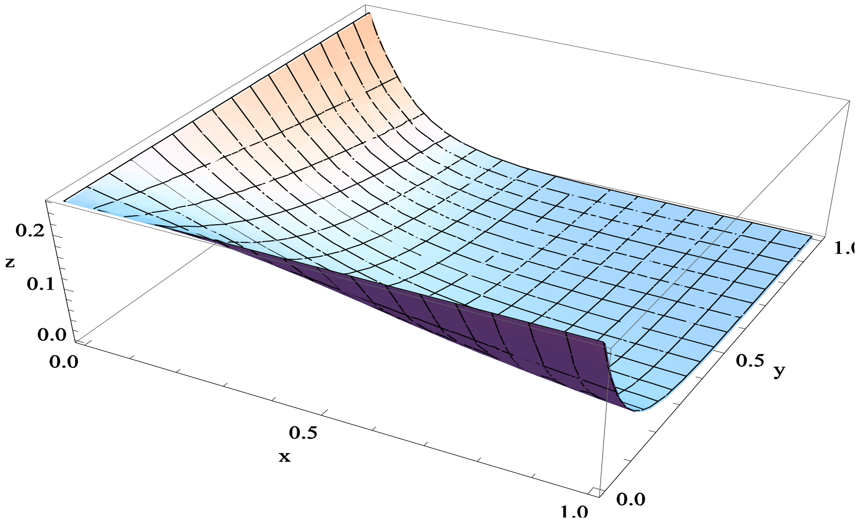

Fig. 5 Spatial temperature profile at $F_0 = 1.0, \mu = 10, \delta = 0.1$.

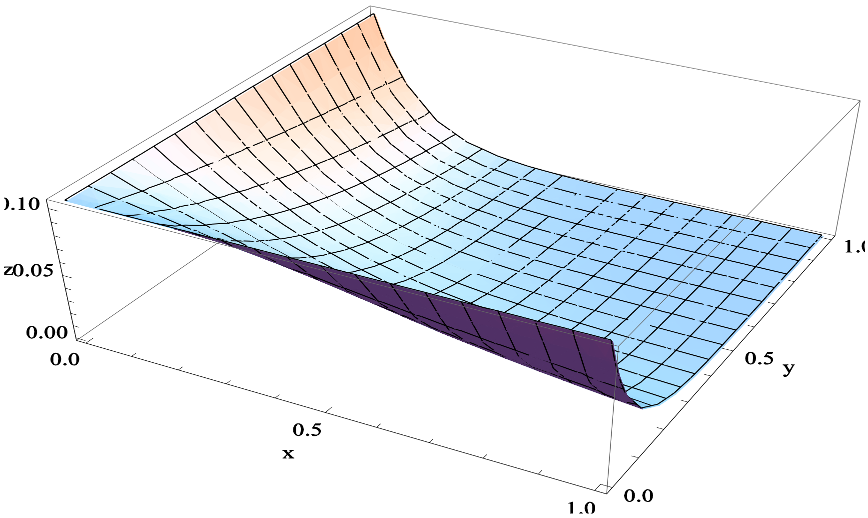

Fig. 6 Spatial temperature profile at $F_0 = 1.0, \mu = 10, \delta = 1.0$.

<u>Case 2:</u> $\theta_2 = 0, \theta_3 = 0, G = e^{-\mu xy} - e^{-\delta F_0}$.

This case is devoted to the effect of internal heat source on the temperature profile. Heat source is modelled as time varying and spatially decaying. The spatial temperature profile for various absorption coefficients (μ) at fixed Fourier number $F_0 = 1.0$ and laser pulse fall-time $(\delta) = 0.5$ is given in Figs. 3-4. Due to the spatially decaying nature of heat source, if we move towards end of both the spatial direction of thin plate, then the amount of heat entered into the body decreases and hence dimensionless temperature decreases with increase of absorption coefficient, as shown in Figs. 3-4.

Figs. 5-6 present the effect of laser pulse fall-time on spatial temperature profile at fixed absorption coefficient and Fourier number. For fixed Fourier number, as laser pulse fall-time increases the amount of heat entered into the body decreases, due to which dimensional temperature into the body decreases.

5. CONCLUSION

The mathematical model describing heat transfer in a thin plate based on single-phase-lagging heat conduction is solved by superposition technique. The solution was obtained by utilizing superposition technique, structure theorem and Fourier series expansion. The effect of Fourier number, absorption coefficient and laser pulse fall time parameter on temperature profile has been observed. The temperature increases with increase of Fourier number and laser pulse fall time parameter but decreases with absorption coefficient.

This technique is very applicable for solving non-homogeneous partial differential equation under most generalized boundary conditions and may be applicable for solving the higher dimensional SPL heat conduction model of general body.

6. NOMENCLATURE

c	Thermal wave propagation speed (m/s)
c_b	Specific heat capacity $(J/kg.K)$
f_r	Reference heat flux $(\mathbf{q}/\mathbf{q}^*)$
F_0	Fourier number $(c^2 t/2\alpha)$
g^*	Internal heat source (W/m^3)
g	Dimensionless heat source $(4\alpha g^*/cf_r)$
k	Thermal conductivity $(W/m.K)$
$\mathbf{q}^*$	Dimensionless heat flux $(\mathbf{q}/f_r)$
$\mathbf{r}$	Position vector
t	Time (s)
T	Temperature (K)
ΔT	Temperature gradient (K/m)
x^*	Spatial coordinate (m)
x	Dimensionless spatial coordinate $(cx^*/2\alpha)$
y^*	Dimensionless spatial coordinate $(cy^*/2\alpha)$
y	Spatial coordinate (m)
α	Thermal diffusivity (m^2/s)
δ^*	Thermal diffusivity $(1/s)$
δ	Dimensionless laser pulse fall-time parameter $(2\tau_q\delta^*)$
θ	Dimensionless Temperature $(kcT/\alpha f_r)$
μ^*	Thermal diffusivity $(1/m)$
μ	Dimensionless absorption coefficient $(2c\tau_q\mu^*)$
ρ	Density (kg/m^3)
τ_q	Phase-lag of heat flux (s)

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