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Necessary Optimality Conditions for Approximate Minima in Unconstrained Finite Minmax Problem

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ABSTRACT: A mathematical optimization is the process of minimizing (or maximizing) a function. The minimum of a function is a critical point and corresponds to gradient (or derivative) of 0. The research work presented in this paper deals with unconstrained minmax problem where the objective function is the maximum of a finite number of smooth convex functions. Obviously, that function is convex but may not be necessarily differentiable. Thus, we can't use gradient method. When gradient information of the objective function is unavailable, unreliable or 'expensive' in terms of computation time, the approximate optimization is ideal. More precisely, we focus on necessary optimality conditions for approximate minima in minmax problem. Firstly, we followed all the details about convex optimization, optimality conditions, subgradient and subdifferential as well as approximate optimization. We present unbiassed and sharp result using standard theorems and references. In here Carathéodory's theorem plays very important role to get our result.
KEYWORDS – approximate, convex, differentiable, optimization, unconstrained

1. INTRODUCTION

Mathematical optimization is the process of minimizing (or maximizing) a function. A mathematical optimization problem or just optimization problem has the form,

$$\text{minimize } f_0(x)$$

$$\text{subject to } f_i(x) \leq b_i \,;\, i = 1,2, \ldots.., m$$

Here the vector $x = (x_1, x_2, \ldots \ldots.., x_m)$ is the optimization variable of the problem, the function $f_0: \mathbb{R}^n \to \mathbb{R}$ is the objective function, the functions $f_i: \mathbb{R}^n \to \mathbb{R}\,; i = 1, 2, 3, \ldots \ldots, m$ are the (inequality) constraint functions and the constants $b_1, b_2, \ldots.., b_n$ are the limits or bounds for the constraints. A vector x^* is called optimal or a solution of the problem, if it has the smallest objective value among all vectors that satisfy the constraints: for any z with $f_1(z) \leq b_1, f_2(z) \leq b_2, \ldots \ldots, f_m(z) \leq b_m$. We have$f_0(z) \geq f_0(x^*)$.

In this paper, we deal with convex unconstrained finite minmax problems which may be categorized as;

$$\text{minimize } f_0(x), \text{ where } f(x) = \min\{f_i(x); i = 1, 2, 3, \ldots.., m\}$$

and each individual $f_i: \mathbb{R}^n \to \mathbb{R}$ are differentiable convex functions. We focus on the finite minmax problem due to its frequent appearance in real – world application. Finite minmax problems occur in numerous applications, such as portfolio optimization, control system design, engineering design and determining the cosine measure of a positive spanning set.

Consider optimization problems of the form;

$$\min f(x), \quad x \in \Omega$$

and for unconstrained optimization problem we have $\Omega = \mathbb{R}^n$. The first question to ask is whether a solution exists for the problem, and it is answered by the well-known Weierstrass theorem: If f is continuous and Ω is compact then a solution a solution for the above problem exists. Let x^* be the point that we take as the maximum. A point that satisfies the necessary condition: $\nabla f(x^*) = 0$ is a stationary point and the sufficient conditions for x^* to be a strict local minimum are $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive definite. Thus, optimization algorithms commonly require gradient calculations. But, in most cases this may not be easy and then we use approximate optimization.

Structurally, a finite minmax problem minimizes the maximum taken over a finite set of functions. We can trivially see that the function f is convex but may not be necessarily differentiable. Therefore, we can't use above necessary and sufficient conditions.

In this paper, we aimed to develop very good and sharp necessary optimality condition for convex unconstrained finite minmax problem using the standard results.

2. LITERATURE REVIEW

One of the first detailed study of optimality conditions for approximate optimization was done by Loridan (1982) where he developed necessary conditions for problems with objective functions which are directionally differentiable. He had also studied the Lagrange multiplier rule for almost $\varepsilon -$ minimum for convex functions.

Hiriart-Urruty (1982) had developed a Lagrangian multiplier rule for $\varepsilon -$ minimization of a convex programming problem in terms of the $\varepsilon -$ subdifferentials of the related functions. Strodiotetal (1983) had also studied Lagrange multiplier rules for the $\varepsilon -$ minimization of convex programming problems in terms of the $\varepsilon -$ subdifferential of functions involved. Very recently Hamel (2001) [6] also developed a Karush-Kuhn-Tucker type condition for a locally convex program in terms of the Clarke subdifferential of the functions involved.

3. METHODOLOGY

The study of approximate optimization is a very beautiful topic in itself. Although we have not chosen to go into details here. Just a few works are in order. Following standard properties and theorems play very important role.

i. A point x^* is a minimizer of a convex function f if and only if f is subdifferentiable at x^* and

$$0 \in \partial f(x^*)$$

i.e. $g = 0$ is a subgradient of f at x^*. This follows directly from the fact that $f(x) \geq f(x^*)$ for all $x \in dom f$.

ii. The subdifferential of the maximum of functions is the convex hull of the union of subdifferentials of the 'active' functions at x.

i.e. $\partial f(x) = Co \cup \{\partial f_i(x) | f_i(x) = f(x)\}$

iii. Consider the problem (MP)

Where $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a convex function and S is a closed and proper subset of $\mathbb{R}^n \times \mathbb{R}^m$. Let $x_0 \in S$ be an $\varepsilon -$ minimum for (MP). Then there exists $\bar{x}$ is also an $\varepsilon -$ minimum for (MP) such that $\| x_0 - \bar{x} \| \leq 1$ and

$$0 \in \partial_\varepsilon f(\bar{x}) + N(S, \bar{x}).$$

If $S = X$ then f can be extended - valued function bounded below and one has

$$0 \in \partial_\varepsilon f(\bar{x}).$$

Proof: Let f be a convex function $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ such that Ω and K are nonempty convex subset of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Assume that K is finite.

Then consider the problem:

$$\min_{x \in \Omega} \left\{ \max_{y \in K} f(x, y) \right\}$$

We know that function f is convex but may not be differentiable. Therefore, we define subdifferential of f at point $\bar{x}$ (set of subgradient of f at the point $\bar{x}$). Then by using above theorem optimality condition for given problem is

$$0 \in \partial_\varepsilon \left\{ \max_{y \in K} f(\bar{x}, y) \right\}. \quad (\bar{x} \text{ is fixed})$$

Property 2 says that,

$$0 \in conv(\{\partial f(\bar{x}, y_i) / i \in I(\bar{x})\})$$

Using the Carathéodory's theorem and standard results of convex optimizations, the necessary optimality condition of minmax problem is

$$0 \in \bigcup \{\partial(\sum_{i \in I(x)} \alpha_i f(\bar{x}, y_i)) | \sum_{i=1}^{n} \alpha_i = 1\}.$$

4. CONCLUSSION

In this study we have proposed a novel necessary optimality condition for convex unconstrained finite minmax problems. This optimality condition has been developed by standard results and references. Considerable future work is available in this research direction. There is considerable future work in the game theory problems, as well as numerous other real-world applications.

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